ALGEBRAIC CURVES
EXERCISE SHEET 9

Unless otherwise specified, k is an algebraically closed field.

Exercise 1.

Let R be a ring and $I, J \subseteq R$ two ideals. I, J are said to be *comaximal* if $I + J = R$.

(1) Show that $IJ \subseteq I \cap J$ for any I, J and that equality holds for comaximal ideals. Can you provide a counter-example when I, J are not assumed to be comaximal?
(2) Suppose I, J are comaximal. Show that, for $m, n \geq 1$ I^m, J^n are also comaximal.

Consider now ideals $I_1, \ldots, I_N \subseteq R$. For $1 \leq i \leq n$, call $J_i = \bigcap_{j \neq i} I_j$.

(3) Suppose that for all i, I_i, J_i are comaximal. Show that for all $n \geq 1$, $I_1^n \cap \ldots \cap I_N^n = (I_1 \ldots I_N)^n = (I_1 \cap \ldots \cap I_N)^n$.

Finally, consider the k-algebra $R = k[x_1, \ldots, x_n]$ and ideals $I, J \subseteq R$.

(4) Show that I, J are comaximal if, and only if, $V(I) \cap V(J) = \emptyset$.

Exercise 2.

Let R be a ring. Recall that a domain is called *integrally closed* if, for any $x \in K = Frac(R)$, if there exist $a_1, \ldots, a_n \in R$ such that $x^n + a_1 x^{n-1} + \ldots + a_n = 0$, then $x \in R$. Show that R is a DVR if, and only if, R is an integrally closed Noetherian local domain with Krull dimension one. (Hint: You can use without proof that any ideal $I \neq (0), R$ in a Noetherian, dimension 1, integrally closed domain can be written uniquely as a product of prime ideals. Can you find a uniformizer of R?)

Exercise 3.

A *valuation* on a field K is a surjective function $\varphi : K \to \mathbb{Z} \cup \{\infty\}$ satisfying the following axioms:

(i) $\varphi(a) = \infty \Leftrightarrow a = 0$
(ii) $\varphi(ab) = \varphi(a) + \varphi(b)$
(iii) $\varphi(a + b) \geq min(\varphi(a), \varphi(b))$

Show that the datum of a DVR with quotient field K is equivalent to the datum of a valuation on K i.e.

(1) Given a valuation φ on K, $R = \{\varphi \geq 0\}$ is a DVR with maximal ideal $\mathfrak{m} = \{\varphi > 0\}$.
(2) Given a DVR R, ord is a valuation on R (assuming we set $ord(0) = \infty$).

Now consider $K = \mathbb{Q}$ and $p \in \mathbb{Z}$ some prime number.

(3) Show that $\mathbb{Z}_{(p)}$ is a DVR. What is the associated valuation ord_p?
(4) Show that any valuation on $\mathbb{Q}$ is equal to ord_p for some prime number p. (Hint: Using Bezout's theorem, you can show that a valuation on $\mathbb{Q}$ is strictly positive in at most one prime.)

Exercise 4.

A simple point P on a curve F with tangent line L at P is called a *flex* if $ord_P^F(L) \geq 3$. The flex is called *ordinary* if $ord_P^F(L) = 3$ and a *higher* flex otherwise.

(1) Let $F = Y - X^n$. For which n does F have a flex at $P = (0,0)$ and what kind of flex?
(2) Suppose that $P = (0,0)$, $L = Y$ and $F = Y + aX^2 + \dots$ (the remaining terms having degree at least 2). Show that P is a flex if, and only if, $a = 0$. Give a simple criterion for calculating $ord_P^F(Y)$.

Exercise 5.

Let $V = V(X^2 - Y^3, Y^2 - Z^3) \subseteq \mathbb{A}_k^3$, $P = (0,0,0)$ and $\mathfrak{m} = \mathfrak{m}_P(V)$. Compute $dim_k(\mathfrak{m}/\mathfrak{m}^2)$.